

SHARP VANISHING ORDER OF SOLUTIONS TO STATIONARY SCHRÖDINGER EQUATIONS ON CARNOT GROUPS OF ARBITRARY STEP

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ABSTRACT. Based on a variant of the frequency function approach of Almgren([Al]), we establish an optimal upper bound on the vanishing order of solutions to stationary Schrödinger equations associated to sub-Laplacian on a Carnot group of arbitrary step \mathbb{G} . Such bound provides a quantitative form of strong unique continuation and can be thought of as an analogue of the recent results of Bakri and Zhu for the standard Laplacian.

1. INTRODUCTION

In this paper we study quantitative uniqueness for zero-order perturbations of sub-Laplacian on Carnot groups. Precisely, we analyze equations of the form

$$(1.1) \quad \Delta_H u = Vu,$$

where Δ_H is the sub-Laplacian associated to a Carnot group \mathbb{G} (see (2.8) below) and the discrepancy E_u (see (2.20)) at the identity e of the solution u satisfies the growth assumption (2.21). The growth assumption (2.21) is to be thought of as the measure of certain symmetry type properties of u and we will have a brief discussion on that in Section 2.

The assumptions on the potential function V are specified in (2.19) below. They represent the counterpart, with respect to the non-isotropic dilations associated with certain horizontal vector fields X_1, \dots, X_N on \mathbb{G} which will be described later in Section 2, of the requirements

$$(1.2) \quad |V(x)| \leq M, \quad | \langle x, DV(x) \rangle | \leq M,$$

for the classical Schrödinger equation $\Delta u = Vu$ in \mathbb{R}^n . To put this paper in the proper historical perspective we recall that for this operator, and under the hypothesis (1.2), quantitative unique continuation results akin to our have been recently obtained in [Bk], by Carleman estimates, and in [Zhu], by means of a variant of Almgren's frequency function introduced in [Ku]. We also refer to [BG1] where a similar result has been established for generalized Baouendi-Grushin operators and [BG2] where sharp quantitative estimates has been established at the boundary of Dini domains for general elliptic equations with Lipschitz principal part. In these papers the authors established sharp estimates on the order of vanishing of solution to Schrödinger equations which generalized those in [DF1] and [DF2] for eigenvalues of the Laplacian on a compact manifold.

Now in the case of Carnot groups, unlike the Euclidean case, the reader will notice that although we have an additional assumption on the discrepancy E_u of the solution u to (1.1) but it is nevertheless not very restrictive in the sense that strong unique continuation property is in general not true for solutions to (1.1). This follows from some interesting work of Bahouri ([Bah]) where the author showed that unique continuation is not true for even smooth and compactly supported perturbations of the sub-Laplacian. Therefore, one cannot expect any quantitative estimates either without further assumptions. On the other hand, it turns out that with precisely this growth assumption on E_u that we have in (2.21), strong unique continuation property (sucp) for (1.1) has been established in [GLa] for the case when $\mathbb{G} = \mathbb{H}^n$ (Heisenberg group which is a Carnot group of step 2). This has been recently generalized to Carnot groups

of arbitrary step in [GR]. It is to be noted that the results in [GLa] and [GR] follow the circle of ideas in the fundamental works [GL1] and [GL2].

Therefore, given the current interest in problems of this type, i.e. quantitative form of strong unique continuation, the purpose of our work is to establish sharp quantitative uniqueness for equations (1.1) in the setup of [GR] where supc has been established, i.e. with the growth assumption on the discrepancy term E_u as in (2.21). Our results should be seen as a subelliptic generalization of the above mentioned Euclidean results in [Bk] and [Zhu]. As the reader will realize, such generalization relies on the deep link existing between the growth properties of a certain generalized Almgren frequency, the sub-elliptic structure on \mathbb{G} and the growth assumption (2.21) on E_u . It turns out that in the end, they beautifully combine. Some of these facts are based on the previous work of Garofalo and Rotz ([GR]) where among several other interesting results, the notion of Almgren's frequency and discrepancy were introduced on a stratified nilpotent Lie group \mathbb{G} (also known as Carnot group).

In this paper, similar to [Zhu], and [BG1], we work with an appropriate weighted version of the Almgren's Frequency which is somewhat different from the one introduced in [GR]. Having said that, we do follow [GR] closely in parts. Since we are interested in the question of sharp vanishing order estimates, it is worth emphasizing that as opposed to Theorem 7.3 in [GR], we need some kind of monotonicity properties of the generalized frequency that is introduced in Section 3 and not just its boundedness (see Theorem 3.1). Moreover, we also need to keep track of the dependence of the several constants that appear in front of the integrals on K (where K is as in (2.19) and can be thought of as the "sub-elliptic" Lipschitz norm of the potential V). Therefore, we have to substantially modify an argument used in the proof of Theorem 7.3 in [GR] which constitutes one of the delicate aspects of this work.

The paper is organized as follows. In Section 2, we introduce the basic notations, gather some crucial preliminary results from [DG], [GR], [GLa] and [GV1] and state our main result. In Section 3, we introduce a monotonicity theorem for the generalized frequency and subsequently prove our main result.

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2. PRELIMINARIES AND STATEMENT OF MAIN RESULT

2.1. Preliminaries. In this section, we first gather some preliminary results as in Section 2 in [GR]. Henceforth in this paper we follow the notations adopted in [GR] with a few exceptions. We recall that a Carnot group of step h is a simply connected Lie group \mathbb{G} whose Lie algebra \mathfrak{g} admits a stratification $\mathfrak{g} = V_1 \oplus \dots \oplus V_h$ which is h nilpotent, i.e., $[V_1, V_j] = V_{j+1}$ for $j = 1, \dots, h-1$ and $[V_j, V_h] = 0$ for $j = 1, \dots, h$. A trivial example is when $\mathbb{G} = \mathbb{R}^n$ and in which case $\mathfrak{g} = V_1 = \mathbb{R}^n$. The simplest non-Abelian example of a Carnot group of step 2 is \mathbb{H}^n , i.e. in \mathbb{R}^{2n+1} , we let $(x, y, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t)$ and the group operation is as follows

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y'))$$

In such a case, we have that V_1 is spanned by

$$(2.1) \quad \begin{aligned} X_i &= \partial_{x_i} + 2y_i \partial_t \\ Y_i &= \partial_{y_i} - 2x_i \partial_t \end{aligned}$$

and V_2 is spanned by ∂_t . We note that the following holds,

$$[X_i, Y_j] = 4\delta_{ij} \partial_t$$

and therefore V_1 generates the whole lie algebra, We would like to mention that over here, we identify the lie algebra \mathfrak{d} with the left invariant vector fields.

Now in a Carnot group \mathbb{G} , by the above assumptions on the Lie algebra, we see that any basis of the horizontal layer V_1 generates the whole \mathfrak{d} . We will respectively denote by

$$(2.2) \quad L_g(g') = gg', \quad R_g(g') = g'g$$

the left and right translation by an element $g \in \mathbb{G}$.

The exponential mapping $\exp : \mathfrak{d} \rightarrow \mathbb{G}$ defines an analytic diffeomorphism onto \mathbb{G} . We recall the Baker-Campbell-Hausdorff formula, see, section 2.15 in [V],

$$(2.3) \quad \exp(c_1)\exp(c_2) = \exp(c_1 + c_2 + \frac{1}{2}[c_1, c_2] + \frac{1}{12}\{[c_1, [c_1, c_2]] - [c_2, [c_1, c_2]]\} + \dots)$$

where the dots indicate commutators of order four and higher. Each element of the layer V_j is assigned a formal degree j . Accordingly, one defines dilations on \mathfrak{d} by the rule

$$(2.4) \quad \Delta_\lambda c = \lambda c_1 + \dots \lambda^h c_h$$

The anisotropic dilations δ_λ on \mathbb{G} are then defined as

$$(2.5) \quad \delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}g$$

Throughout the paper, we will indicate by dg the bi-invariant Haar measure on \mathbb{G} obtained by lifting via the exponential map \exp the Lebesgue measure on \mathbb{G} . Let $m_j = \dim V_j$. One can check that

$$(2.6) \quad (d \circ \delta_\lambda)(g) = \lambda^Q dg$$

where $Q = \sum_{j=1}^h j m_j$. Q is referred to as the homogeneous dimension of \mathbb{G} and is in general different from the topological dimension of \mathbb{G} which is $\sum_{j=1}^h m_j$.

Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of the first layer V_1 of the Lie algebra. We define the corresponding left invariant smooth vector fields by the formula

$$(2.7) \quad X_i(g) = dL_g(e_i), \quad i = 1, \dots, m$$

where dL_g denote the differential of L_g . We assume that \mathbb{G} is endowed with a left-invariant Riemannian metric such that $\{X_1, \dots, X_m\}$ are orthonormal. The corresponding subLaplacian is defined by the formula

$$(2.8) \quad \Delta_H u = \sum_{i=1}^m X_i^2 u$$

We note that by Hormander's theorem, Δ_H is hypoelliptic. We with indicate with e the identity element of \mathbb{G} .

Let $\Gamma(g, g') = \Gamma(g', g)$ be the positive fundamental solution of $-\Delta_H$. It turns out that Γ is left invariant, i.e.,

$$(2.9) \quad \Gamma(g, g') = \tilde{\Gamma}(g^{-1} \circ g')$$

For every $r > 0$, let

$$(2.10) \quad B_r = \{g \in \mathbb{G} | \Gamma(g, e) > \frac{1}{r^{Q-2}}\}$$

It was proved by Folland [F1] that $\tilde{\Gamma}(g)$ is homogeneous of order $2 - Q$ with respect to the non-isotropic dilations (2.5). Therefore, if we define

$$(2.11) \quad \rho(r) = \Gamma(g)^{\frac{-1}{Q-2}}$$

then ρ is homogenous of degree 1. One can immediately see that B_r can be equivalently characterized as

$$(2.12) \quad B_r = \{\rho(g) < r\}$$

We let $S_r = \partial B_r$. We note that since Γ is homogeneous of degree $2 - Q$, we have

$$(2.13) \quad Z\Gamma = (2 - Q)\Gamma$$

Now by strong maximum principle (Since $\Gamma(g, e)$ is harmonic for $g \neq e$), we have that $\Gamma(g, e) > 0$ for all $g \neq e$. Now since $Z\Gamma = \langle D\Gamma, Z \rangle$, where $D\Gamma$ is the Riemannian gradient with respect to the left invariant metric, we conclude from (2.13) that $D\Gamma$ never vanishes. Therefore, by implicit function theorem, we conclude that the level sets S_r are smooth hypersurfaces in \mathbb{G} .

The position $d(g, g')$ defined by

$$(2.14) \quad d(g, g') = \rho(g^{-1} \circ g')$$

defines a pseudo-distance on \mathbb{G} . We let Z to be the smooth vector field which corresponds to the infinitesimal generator of the non-isotropic dilations (2.5). Note that Z is characterized by the following property

$$(2.15) \quad \frac{d}{dr}u(\delta_r g) = \frac{1}{r}Zu(\delta_r g)$$

Therefore if u is homogenous of degree k with respect to (2.5), i.e., $u(\delta_r g) = r^k u(g)$, then we have that $Zu = ku$. In what follows, we denote by

$$(2.16) \quad \nabla_H u = \sum_{i=1}^m X_i u X_i$$

the horizontal gradient of u . We also let

$$(2.17) \quad |\nabla_H u|^2 = \sum (X_i u)^2$$

2.2. Statement of the main result. Let u be a solution to

$$(2.18) \quad \Delta_H u = Vu$$

We will assume apriori that $u, X_i u, X_i X_j u$ exist classically and are continuous in B_1 . We will assume that the potential V satisfies

$$(2.19) \quad |V| \leq K|\nabla_H \rho|^2 \quad |ZV| \leq K|\nabla_H \rho|^2$$

for some $K > 1$.

As in [GR], we define the discrepancy E_u at e by

$$(2.20) \quad E_u = \langle \nabla_H u, \nabla_H \rho \rangle - \frac{Zu}{\rho} |\nabla_H \rho|^2$$

Similar to [GR], we will also assume that

$$(2.21) \quad |E_u| \leq \frac{f(\rho)}{\rho} |\nabla_H \rho|^2 |u|$$

where $f : (0, 1) \rightarrow (0, \infty)$ is a continuous increasing function which satisfies the Dini integrability condition

$$(2.22) \quad \int_0^1 \frac{f(t)}{t} dt < K_0, \quad |f| \leq K_1$$

We now list a few classes of examples from [GR] in which the assumption (2.21) holds.

When u is radial, i.e., if $u(g) = f(\rho(g))$, then it follows from a straightforward calculation (See Proposition 9.6 in [GR]) that $E_u \equiv 0$.

Also, if we specialize to the case when $\mathbb{G} = \mathbb{H}^n$ and when u is polyradial, i.e. with $g = (w_1, \dots, w_n, t)$ where $w_i = (x_i, y_i)$, we have that $u(g) = \phi(|w_1|, \dots, |w_n|, t)$, then $E_u \equiv 0$. (see

Proposition 9.11 in [GR]). It is however not true that for general groups of Heisenberg type (see Section 9.1 in [GR] for the precise definition of groups of Heisenberg type), polyradial functions have zero discrepancy. (see Section 9 in [GR] for a counterexample) Nevertheless, given these two classes of examples, we would like to think of the growth condition (2.21) on E_u as the measure of certain symmetry type properties of u .

We now state our main result.

Theorem 2.1. *Let u be a solution to (2.18) in B_1 such that $|u| \leq C_0$ which additionally satisfies (2.21). Let V satisfy (2.19). Then there exists a universal $a \in (0, 1/3)$, and constants C_1, C_2 depending on Q, C_0 and K_0, K_1 in (2.22) and also $\int_{B_{1/3}} u^2 |\nabla_H \rho|^2 dg$ such that for all $0 < r < \frac{a}{3}$, one has*

$$(2.23) \quad \|u\|_{L^\infty(B_r)} \geq C_1 r^{C_2 \sqrt{K}}$$

Remark 2.2. *We note that in the elliptic case as in [BG2], [Zhu], we have that $|\nabla_H \rho|^2 \equiv 1$ and in such a case, the constant K in (2.19) can be taken to be $C(\|V\|_{W^{1,\infty}} + 1)$ for some universal C . We thus see that in the elliptic case, (2.23) reduces to the Euclidean result in [Bk] and [Zhu]. Therefore, Theorem 2.1 can be thought of as a subelliptic analogue of the sharp quantitative uniqueness result for the standard Laplacian.*

Remark 2.3. *It is also worth mentioning as well that when $\mathbb{G} = \mathbb{H}^n$ and $E_u \equiv 0$, then from Proposition 9.8 in [GR], it follows that u solves the Schrodinger equation corresponding to the Baouendi-Grushin operator*

$$(2.24) \quad \Delta_z u + \frac{|z|^2}{4t} \Delta_t u = V u$$

for which the sharp quantitative estimate (2.23) follows from Theorem 1.1 in [BG1]. The Baouendi Grushin operator was first introduced in [Ba]. We would also like to refer to [Gr1], [Gr2] where hypoelliptic results for generalized Baouendi-Grushin operators have been proved.

3. PROOF OF THEOREM 2.1

3.1. Monotonicity of a generalized frequency. Following [Zhu] and [BG1], for $\alpha > 0$ to be decided later, we let

$$(3.1) \quad H(r) = \int_{B_r} u^2 |\nabla_H \rho|^2 (r^2 - \rho^2)^\alpha dg$$

We would like to mention that in the context of Almgren's frequency, in the Euclidean case such weights first appeared in [Ku]. For notational convenience, we let $|\nabla_H \rho|^2 = \psi$. Therefore with this new notation, we have that

$$(3.2) \quad H(r) = \int_{B_r} u^2 (r^2 - \rho^2)^\alpha \psi$$

By differentiating with respect to r , we get that

$$(3.3) \quad H'(r) = 2\alpha r \int u^2 (r^2 - \rho^2)^{\alpha-1} \psi$$

Using the identity

$$(3.4) \quad (r^2 - \rho^2)^{\alpha-1} = \frac{1}{r^2} (r^2 - \rho^2)^\alpha + \frac{\rho^2}{r^2} (r^2 - \rho^2)^{\alpha-1}$$

the latter equation can be rewritten as

$$(3.5) \quad H'(r) = \frac{2\alpha}{r} H(r) + \frac{2\alpha}{r} \int u^2 (r^2 - \rho^2)^{\alpha-1} \rho^2 \psi$$

Now since ρ has homogeneity 1, we have that $Z\rho = \rho$. Hence $(r^2 - \rho^2)^{\alpha-1}\rho^2$ can be rewritten as

$$(3.6) \quad (r^2 - \rho^2)^{\alpha-1}\rho^2 = -\frac{1}{2\alpha}Z(r^2 - \rho^2)^\alpha$$

Therefore we get that

$$(3.7) \quad H'(r) = \frac{2\alpha}{r} - \frac{1}{r} \int u^2 Z(r^2 - \rho^2)^\alpha \psi$$

Now we note that since ρ is homogeneous of degree 1, hence $|\nabla_H \rho|^2 = \psi$ is homogeneous of degree 0. Therefore we have that

$$(3.8) \quad Z\psi(g) = 0$$

for $g \neq e$. We also note that

$$(3.9) \quad \text{div}_{\mathbb{G}} Z = Q$$

(see for instance [DG]) Note that over here, $\text{div}_{\mathbb{G}}$ denotes the Riemmanian divergence on \mathbb{G} . Now by using the Divergence theorem on \mathbb{G} with respect to its Riemmanian structure and also by using (3.8), (3.9), we get that

$$(3.10) \quad H'(r) = \frac{2\alpha + Q}{r} H(r) + \frac{2}{r} \int u Z u (r^2 - \rho^2)^\alpha \psi$$

Over here, we crucially use the fact that since $|\nabla_H \rho|^2$ has homogeneity 0, therefore it is bounded and hence the integration by parts can be justified by an approximation type argument. Now by using (2.21), we get that

$$(3.11) \quad H'(r) = \frac{2\alpha + Q}{r} H(r) + \frac{2}{r} \int u \rho \langle \nabla_H u, \nabla_H \rho \rangle (r^2 - \rho^2)^\alpha + K(r)$$

where

$$(3.12) \quad |K(r)| \leq \frac{f(r)}{r} H(r)$$

(3.11) can be rewritten as

$$(3.13) \quad H'(r) = \frac{2\alpha + Q}{r} H(r) + \frac{1}{(\alpha + 1)r} I(r) + K(r)$$

where

$$(3.14) \quad I(r) = 2(\alpha + 1) \int u \langle \nabla_H u, \nabla_H \rho \rangle (r^2 - \rho^2)^\alpha \rho = - \int u \langle \nabla_H u, \nabla_H (r^2 - \rho^2)^{\alpha+1} \rangle$$

Now we note that the following identity holds(see for instance [GV1])

$$(3.15) \quad \text{div}_{\mathbb{G}} X_i = 0$$

Therefore, by applying integrating by parts to (3.14) and by using the equation (2.18) and the identity (3.15) we get that,

$$(3.16) \quad I(r) = \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1} + V u^2 (r^2 - \rho^2)^{\alpha+1}$$

We now define the generalized frequency of u as

$$(3.17) \quad N(r) = \frac{I(r)}{H(r)}$$

The central result of this section which implies (2.23) in Theorem 2.1 is the following monotonicity result of $N(r)$.

Theorem 3.1. For $\alpha = \sqrt{K}$, we have that there exists universal C_1, C_2 depending on Q, K_0 such that

$$(3.18) \quad r \rightarrow e^{C_1 \int_0^r \frac{f(t)}{t}} (N(r) + C_2 K(r^2 + \int_0^r \frac{f(t)}{t} dt))$$

is monotone increasing on $(0, 1)$.

Proof. The proof will be divided into several steps. We first calculate $I'(r)$. By differentiating the expression in (3.16) with respect to r , we get that

$$(3.19) \quad I'(r) = 2(\alpha + 1)r \int |\nabla_H u|^2 (r^2 - \rho^2)^\alpha + 2(\alpha + 1)r \int V u^2 (r^2 - \rho^2)^\alpha$$

This can be rewritten as

$$(3.20) \quad I'(r) = \frac{2(\alpha + 1)}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1} + \frac{2(\alpha + 1)}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^\alpha \rho^2 + 2(\alpha + 1)r \int V u^2 (r^2 - \rho^2)^\alpha$$

Using the fact that $Z\rho = \rho$, the second term on the right hand side of above expression can be rewritten as

$$(3.21) \quad \frac{2(\alpha + 1)}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^\alpha \rho^2 = -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha+1}$$

At this point, we need the following Rellich type identity established in corollary 3.3 in [GV1] which can be thought of as the sub-elliptic analogue of Rellich type identity established in [PW]. For a C^1 vector field F , we have that

$$(3.22) \quad \begin{aligned} & 2 \int_{\partial B_r} F u < \nabla_H u, N_H > dH^{n-1} + \int_{B_r} \operatorname{div}_{\mathbb{G}} F |\nabla_H u|^2 dg \\ & - 2 \int_{B_r} X_i u [X_i, F] u dg - 2 \int_{B_r} F u \Delta_{\mathbb{H}} u dg \\ & = \int_{\partial B_r} |\nabla_H u|^2 < F, \nu > dH^{n-1} \end{aligned}$$

We now apply the identity (3.22) to the vector field $F = (r^2 - \rho^2)^{\alpha+1} Z$. We note that the boundary terms don't appear due to the presence of the weight $(r^2 - \rho^2)^{\alpha+1}$. Therefore we get,

$$(3.23) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z (r^2 - \rho^2)^{\alpha+1} = -\frac{1}{r} \int |\nabla_H u|^2 \operatorname{div}_{\mathbb{G}}(F) + \frac{Q}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1}$$

where we used the fact that $\operatorname{div}_{\mathbb{G}} Z = Q$. Now by applying (3.22), we get that

$$(3.24) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z (r^2 - \rho^2)^{\alpha+1} = -\frac{2}{r} \int X_i u [X_i, F] u dg - \frac{2}{r} \int F u \Delta_H u dg + \frac{Q}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1}$$

Now we note that the following holds

$$(3.25) \quad [X_i, Z] = X_i$$

See for instance [DG]. Therefore by using (3.25), we have that

$$(3.26) \quad [X_i, F] u = X_i (r^2 - \rho^2)^{\alpha+1} Z + (r^2 - \rho^2)^{\alpha+1} X_i = -2(\alpha + 1) \rho (r^2 - \rho^2)^\alpha X_i \rho Z + (r^2 - \rho^2)^{\alpha+1} X_i$$

By using (3.26) in (3.24) we get that,

$$(3.27) \quad \begin{aligned} & -\frac{1}{r} \int |\nabla_H u|^2 Z (r^2 - \rho^2)^{\alpha+1} = \frac{4(\alpha + 1)}{r} \int < \nabla_H u, \nabla_H \rho > \rho Z u (r^2 - \rho^2)^\alpha \\ & - \frac{2}{r} \int V u Z u (r^2 - \rho^2)^{\alpha+1} + \frac{Q - 2}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1} \end{aligned}$$

Now by using the assumption (2.21) we get that

$$(3.28) \quad -\frac{1}{r} \int |\nabla_H u|^2 Z(r^2 - \rho^2)^{\alpha+1} = \frac{4(\alpha+1)}{r} \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi + \frac{Q-2}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1} - \frac{2}{r} \int VuZu(r^2 - \rho^2)^{\alpha+1} + K_1(r)$$

where

$$(3.29) \quad |K_1(r)| \leq \frac{4(\alpha+1)f(r)}{r} \int (r^2 - \rho^2)^\alpha |u| |Zu| \psi$$

Therefore by substituting the above expression in (3.20) we get that,

$$(3.30) \quad I'(r) = \frac{2\alpha+Q}{r} \int |\nabla_H u|^2 (r^2 - \rho^2)^{\alpha+1} + \frac{4(\alpha+1)}{r} \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi + 2(\alpha+1)r \int Vu^2 (r^2 - \rho^2)^\alpha - \frac{2}{r} \int VuZu(r^2 - \rho^2)^{\alpha+1} + K_1(r)$$

Recalling the definition of $I(r)$, we can write the above as

$$(3.31) \quad I'(r) = \frac{2\alpha+Q}{r} I(r) - \frac{2\alpha+Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha+1} + \frac{4(\alpha+1)}{r} \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi + 2(\alpha+1)r \int Vu^2 (r^2 - \rho^2)^\alpha - \frac{2}{r} \int VuZu(r^2 - \rho^2)^{\alpha+1} + K_1(r)$$

Now by integrating by parts and using the fact that $\operatorname{div}_{\mathbb{G}} Z = Q$, we get that

$$(3.32) \quad -\frac{2}{r} \int VuZu(r^2 - \rho^2)^{\alpha+1} = -\frac{1}{r} \int Z(u^2)V(r^2 - \rho^2)^{\alpha+1}$$

$$(3.33) \quad = \frac{1}{r} \int u^2 \operatorname{div}_{\mathbb{G}}((r^2 - \rho^2)^{\alpha+1} VZ) = \frac{Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha+1} + \frac{1}{r} \int u^2 ZV(r^2 - \rho^2)^{\alpha+1} - \frac{2(\alpha+1)}{r} \int Vu^2 \rho^2 (r^2 - \rho^2)^\alpha$$

Now we note that by (2.19)

$$(3.34) \quad \left| \frac{Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha+1} \right| \leq CKrH(r)$$

and also

$$(3.35) \quad \left| \frac{1}{r} \int u^2 ZV(r^2 - \rho^2)^{\alpha+1} \right| \leq CKrH(r)$$

some universal C . We now write the expression $\frac{2\alpha+Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha+1}$ as

$$(3.36) \quad \frac{2\alpha+Q}{r} \int Vu^2 (r^2 - \rho^2)^{\alpha+1} = (2\alpha+Q)r \int Vu^2 (r^2 - \rho^2)^\alpha - \frac{2\alpha+Q}{r} \int Vu^2 (r^2 - \rho^2)^\alpha \rho^2$$

Therefore, by using (3.32), (3.34), (3.35) and (3.36) in (3.31) and also (2.19) we get that

$$(3.37) \quad I'(r) = \frac{2\alpha+Q}{r} I(r) + \frac{4(\alpha+1)}{r} \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi + O(1)KrH(r) + K_1(r)$$

Now from (2.21) it follows that

$$(3.38) \quad \int uZu(r^2 - \rho^2)^\alpha \psi = \int u \langle \nabla_H u, \nabla_H \rho \rangle \rho(r^2 - \rho^2)^\alpha + O(f(r)H(r))$$

Therefore from the definition of $N(r)$ as in (3.17), from (3.13), (3.37) and also by using (3.38) we get that

$$\begin{aligned}
 (3.39) \quad N'(r) &= \frac{I'(r)}{H(r)} - \frac{H'(r)}{H(r)}N(r) \\
 &\geq -C_1 \frac{f(r)}{r}N(r) - C_2 Kr \\
 &\quad + \left(\frac{4(\alpha+1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{rH(r)} - \frac{4(\alpha+1) (\int (r^2 - \rho^2)^\alpha uZu\psi)^2}{rH(r)^2} \right) \\
 &\quad - \frac{8(\alpha+1)f(r) (\int |u| |Zu| (r^2 - \rho^2)^\alpha \psi)}{rH(r)}
 \end{aligned}$$

where C_1, C_2 are universal. At this point, we need an adapted form of the argument used in Section 7 in [GR]. Before proceeding further, we make the following remark.

Remark 3.2. *In the subsequent expressions, all the constants C_i 's, \tilde{C}_i 's that appear are universal and only depends on C_0, Q and K_0 in (2.22).*

We first note that

$$(3.40) \quad \int uZu(r^2 - \rho^2)^\alpha \psi = \frac{I(r)}{2(\alpha+1)} + H_1(r)$$

where

$$(3.41) \quad |H_1(r)| \leq f(r)H(r)$$

Now from the representation of $I(r)$ as in (3.16) and from the assumption on V as in (2.19), we note that

$$(3.42) \quad I(r) + Kr^2H(r) \geq 0$$

Therefore, since $\alpha = \sqrt{K}$, we get that

$$(3.43) \quad \frac{I(r)}{2(\alpha+1)} + \sqrt{K}r^2H(r) \geq 0$$

By substituting (3.43) in (3.40), we get that

$$(3.44) \quad \int uZu(r^2 - \rho^2)^\alpha \psi + \sqrt{K}r^2H(r) \geq -f(r)H(r)$$

which can be rewritten as

$$(3.45) \quad \int uZu(r^2 - \rho^2)^\alpha \psi + \sqrt{K}r^2H(r) + f(r)H(r) \geq 0$$

We now distinguish 2 cases. Either we have that

$$(3.46) \quad \left(\int u^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \left(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \leq \sqrt{2} \left(\int uZu(r^2 - \rho^2)^\alpha \psi + 8\sqrt{K}r^2H(r) + f(r)H(r) \right)$$

or

$$(3.47) \quad \left(\int u^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \left(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} > \sqrt{2} \left(\int uZu(r^2 - \rho^2)^\alpha \psi + 8\sqrt{K}r^2H(r) + f(r)H(r) \right)$$

If (3.46) occurs, then by applying Cauchy-Schwartz inequality to the expression

$$\left(\frac{4(\alpha+1) \int (Zu)^2(r^2 - \rho^2)^\alpha \psi}{rH(r)} - \frac{4(\alpha+1) (\int (r^2 - \rho^2)^\alpha uZu\psi)^2}{rH(r)^2} \right)$$

in (3.39), we see that the expression is non-negative.

Now we estimate the term $-\frac{8(\alpha+1)f(r)(\int |u||Zu|(r^2-\rho^2)^\alpha\psi)}{rH(r)}$ in (3.39) by Cauchy-Schwartz and the estimate (3.46) and thereby obtain

$$(3.48) \quad \begin{aligned} & \left| \frac{8(\alpha+1)f(r)(\int |u||Zu|(r^2-\rho^2)^\alpha\psi)}{rH(r)} \right| \\ & \leq \frac{8(\alpha+1)f(r)(\int uZu(r^2-\rho^2)^\alpha\psi + 8\sqrt{K}r^2H(r) + f(r)H(r))}{rH(r)} \end{aligned}$$

Now from (3.38) and the definition of $I(r)$ we get that

$$(3.49) \quad \begin{aligned} N'(r) & \geq -C_1 \frac{f(r)}{r} N(r) - C_2 Kr \\ & \quad - \tilde{C} K \frac{f^2(r)}{r} - \tilde{C}_1 K \frac{f(r)}{r} \end{aligned}$$

In (3.49), we crucially used the fact that $\alpha = \sqrt{K} \leq K$. Now since $|f| \leq K_1$ we therefore obtain that

$$(3.50) \quad N'(r) \geq -C_1 \frac{f(r)}{r} N(r) - C_2 Kr - C_6 K \frac{f(r)}{r}$$

If instead (3.47) occurs, then there are 2 sub-cases. Either

$$(3.51) \quad \int uZu(r^2-\rho^2)^\alpha\psi \geq 0$$

or

$$(3.52) \quad \int uZu(r^2-\rho^2)^\alpha\psi \leq 0$$

If (3.51) occurs, then by using the inequality $(a+b)^2 \geq a^2$ when $a, b \geq 0$, we get that

$$(3.53) \quad \left(\int u^2(r^2-\rho^2)^\alpha\psi \right) \left(\int (Zu)^2(r^2-\rho^2)^\alpha\psi \right) \geq 2 \left(\int uZu(r^2-\rho^2)^\alpha\psi \right)^2$$

From (3.53), it follows that

$$(3.54) \quad \frac{4(\alpha+1) \int (Zu)^2(r^2-\rho^2)^\alpha\psi}{rH(r)} - \frac{4(\alpha+1) \left(\int (r^2-\rho^2)^\alpha uZu \right)^2}{rH(r)^2}$$

$$(3.55) \quad \geq \frac{2(\alpha+1) \int (Zu)^2(r^2-\rho^2)^\alpha\psi}{rH(r)}$$

By using the above inequality in (3.39), we get that

$$(3.56) \quad \begin{aligned} N'(r) & \geq -C_1 \frac{f(r)}{r} N(r) - C_2 Kr \\ & \quad + \frac{2(\alpha+1) \int (Zu)^2(r^2-\rho^2)^\alpha\psi}{rH(r)} - \frac{8(\alpha+1)f(r)(\int |u||Zu|(r^2-\rho^2)^\alpha\psi)}{rH(r)} \end{aligned}$$

Now by applying Cauchy-Schwartz inequality with ε , i.e. the inequality

$$(3.57) \quad ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$$

to the term $\frac{8(\alpha+1)f(r)(\int |u||Zu|(r^2-\rho^2)^\alpha\psi)}{rH(r)}$ in (3.56) for small enough ε , we get that

$$(3.58) \quad N'(r) \geq -C_1 \frac{f(r)}{r} N(r) - C_2 Kr - C_8 K \frac{f(r)}{r}$$

where we also used that $|f| \leq K_1$. If instead (3.52) occurs, then we note that since (3.45) holds, i.e.

$$(3.59) \quad \int uZu(r^2 - \rho^2)^\alpha \psi + \sqrt{K}r^2 H(r) + f(r)H(r) \geq 0$$

Hence the following inequality holds

$$(3.60) \quad \int uZu(r^2 - \rho^2)^\alpha \psi + 8\sqrt{K}r^2 H(r) + f(r)H(r) \geq 7\sqrt{K}r^2 H(r)$$

Therefore we have that from (3.47) the following holds

$$(3.61) \quad \left(\int u^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \left(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \geq 7\sqrt{2}\sqrt{K}r^2 H(r)$$

Squaring the above inequality and by cancelling off $H(r)$ from both sides gives

$$(3.62) \quad \left(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi \right) \geq 94Kr^4 H(r)$$

Again (3.47) trivially implies that

$$(3.63) \quad \left(\int u^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} \left(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi \right)^{1/2} > \sqrt{2} \left(\int uZu(r^2 - \rho^2)^\alpha \psi + \sqrt{K}r^2 H(r) + f(r)H(r) \right)$$

Squaring the above inequality (where we taking into account that the right hand side in the above inequality is non-negative due to (3.45)) and then by using $(a+b)^2 \geq a^2 + 2ab$ for $b \geq 0$ on the right hand side of the corresponding squared inequality, we get that the following holds,

$$(3.64) \quad \begin{aligned} H(r) \left(\int (Zu)^2(r^2 - \rho^2)^\alpha \psi \right) &\geq 2 \left(\int uZu(r^2 - \rho^2)^\alpha \psi \right)^2 \\ &+ 4\sqrt{K}r^2 \left(\int uZu(r^2 - \rho^2)^\alpha \psi \right) H(r) \\ &+ 4f(r) \left(\int uZu(r^2 - \rho^2)^\alpha \psi \right) H(r) \end{aligned}$$

Now we note that (3.45) and (3.52) implies that

$$(3.65) \quad \left| \int uZu(r^2 - \rho^2)^\alpha \psi \right| \leq \sqrt{K}r^2 H(r) + f(r)H(r)$$

Therefore by using (3.64) in (3.39) and then subsequently the estimate (3.65), we get that,

$$(3.66) \quad \begin{aligned} N'(r) &\geq -C_1 \frac{f(r)}{r} N(r) - C_2 Kr \\ &+ \frac{2(\alpha+1) \int (Zu)^2(r^2 - \rho^2)^\alpha \psi}{rH(r)} - 16K^{3/2}r^3 \\ &- C_{10}K \frac{f(r)}{r} - \frac{8(\alpha+1)f(r) \left(\int |u| |Zu| (r^2 - \rho^2)^\alpha \psi \right)}{rH(r)} \end{aligned}$$

where we have also used the fact that $|f| \leq K_1$ and $\alpha = \sqrt{K}$. We have also used the fact that $(\alpha+1) \leq 2\alpha$ since $\alpha > 1$. Now we note that from (3.62) the following holds

$$(3.67) \quad \frac{(\alpha+1) \int (Zu)^2(r^2 - \rho^2)^\alpha \psi}{rH(r)} \geq 94K^{3/2}r^3$$

Therefore by using (3.67) in (3.66), we get that

$$(3.68) \quad \begin{aligned} N'(r) &\geq -C_1 \frac{f(r)}{r} N(r) - C_2 K r - C_{10} K \frac{f(r)}{r} \\ &\quad + \frac{(\alpha + 1) \int (Zu)^2 (r^2 - \rho^2)^\alpha \psi}{r H(r)} \\ &\quad - \frac{8(\alpha + 1) f(r) (\int |u| |Zu| (r^2 - \rho^2)^\alpha \psi)}{r H(r)} \end{aligned}$$

Again by applying Cauchy-Schwartz inequality with ε to the term $-\frac{8(\alpha+1)f(r)(\int |u||Zu|(r^2-\rho^2)^\alpha \psi)}{rH(r)}$, we conclude that (3.58) holds or equivalently (3.50) holds as in the case (3.46). Estimate (3.50)/(3.58) implies (3.19) in a standard way. \square

3.2. Proof of estimate (2.23) in Theorem 2.1. We note that although (3.19) in the monotonicity Theorem 3.1 is different from its counterpart Theorem 3.1 in [BG1], nevertheless it still implies that the following inequality holds

$$(3.69) \quad N(r) \leq \overline{C}(N(s) + C_2 K), \quad \text{for } 0 < r < s < 1.$$

Using (3.69), we can argue in the same way as in Section 4 in [BG1] to conclude that our desired estimate (2.23) in Theorem 2.1 holds.

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